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Classical and relativistic dynamics of supersolids: variational principle

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Abstract

We present a phenomenological Lagrangian and Poisson brackets for obtaining nondissipative hydrodynamic theory of supersolids. A Lagrangian is constructed on the basis of unification of the principles of non-equilibrium thermodynamics and classical field theory. The Poisson brackets, governing the dynamics of supersolids, are uniquely determined by the invariance requirement of the kinematic part of the found Lagrangian. The generalization of Lagrangian is discussed to include the dynamics of vortices. The obtained equations of motion do not account for any dynamic symmetry associated with Galilean or Lorentz invariance. They can be reduced to the original Andreev–Lifshitz equations to require Galilean invariance. We also present a relativistic-invariant supersolid hydrodynamics, which might be useful in astrophysical applications.

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1. Introduction

A supersolid state of matter was theoretically predicted by Andreev and Lifshitz in 1969 [1]. They noted that in crystals with a large amplitude of zero-point motion (a large value of the de Boehr parameter [2]) exceptional quantum-mechanical effects can occur. In particular, the number of lattice sites in such crystals is not equal to the number of atoms and vacancies exist even at the absolute zero temperature. These vacancies are caused by zero-point energy, which also causes them to be mobile as waves. Since the vacancies are delocalized, they can be considered as weakly interacting quasiparticles. In a cloud of quasiparticles, there may occur a phase transition to the Bose condensed state. Therefore, a supersolid is a state with simultaneously broken continuous translational and global $U(1)$ symmetries and, therefore, it has both crystalline and superfluid order. Later on, the possibility of the

supersolid phase in ^{4}He was also discussed by Chester [3] and Leggett [4]. For many years, in spite of numerous experimental efforts [5], the supersolid behavior of ^{4}He was not discovered.

However, in 2004, Kim and Chan reported the possible observation of superfluidity in solid ^{4}He. The superfluid density ρ_s was observed at temperatures below about 200 mK in a torsional oscillator containing both bulk solid helium [7] and a porous Vycor matrix with it [6]. The occurrence of the superfluid density in a solid helium was accompanied by nonclassical rotational inertia signal. Soon, several independent groups confirmed the evidence of nonclassical rotational inertia and the possible existence of the supersolid phase [8–11]. At the same time the low temperature measurements found no evidence for superflow in solid ^{4}He [12, 13] and some experiments [8, 9] showed the absence of the supersolid behavior in annealed solid ^{4}He samples. However, the most recent experiment showed the superfluid-like mass transport through a region with solid hcp ^{4}He off the melting curve [14]. Though the microscopic nature of supersolidity is still unclear, the experimental results on annealing confirm the recent path integral Monte Carlo simulations [15–18] that ideal crystals do not exhibit the superfluid properties. In addition, the simulations predict that there are no vacancies in the ground state of solid ^{4}He. Note that some microscopic models for supersolids, in which the lattice parameters are changed independently (the lattice spacing is an independent thermodynamic variable), do not account for the presence of defects [19, 20].

The hydrodynamic theory of supersolids was studied by a number of authors [1, 21, 22] within the phenomenological approach and in close analogy to the two-fluid model. The microscopic derivation of the corresponding hydrodynamic equations was given in [23] on the basis of the Gibbs statistical operator and Bogolyubov's method of quasiaverages [24, 25]. There also exists the useful Poisson bracket formalism to hydrodynamic theory of various complex systems, including liquid crystals [26–29], superfluid liquids [26, 30–36], (anti)ferromagnets [35, 37], spin glasses [26], etc. Recently, an effective Lagrangian describing the low-energy dynamics of supersolids has been studied [38].

This paper concerns a phenomenological Lagrangian and the Poisson bracket formalism leading to nondissipative hydrodynamics of supersolids. The main advantage of this formalism is that it gives the straightforward derivation of dynamic equations once the dynamic variables are defined and the Poisson brackets are found. It also provides a tool for studying Lyapunov stability of equilibrium solutions [39]. Hydrodynamics of supersolids is entirely specified both by the conserved quantities such as the densities of momentum, entropy, particle number and by the fields related to the broken continuous translational and global $U(1)$ symmetries. In order to construct a phenomenological Lagrangian describing the low-frequency dynamics of supersolids in terms of the mentioned hydrodynamic fields, we also have to introduce a non-physical (cyclic) dynamic variable, conjugate to entropy [35]. Using the electromagnetic analogy with superconductors [31], in which the presence of vortices is related to the existence of a magnetic field, we show how to modify the obtained Lagrangian so that it reproduces the dynamics with vortices. In our approach, the Poisson brackets of hydrodynamic variables are uniquely determined by the invariance requirement of the kinematic part of constructed Lagrangian. The corresponding hydrodynamic equations do not account for any dynamic symmetry associated with Galilean or Lorentz invariance. The constraint on the thermodynamic potential density, following from the Galilean invariance, leads to Andreev–Lifshitz hydrodynamics. We also derive the relativistic-invariant hydrodynamics of supersolids that might be useful for describing the crystalline superfluidity in compact stars [40]. The limiting cases of normal and superfluid liquids as well as the elasticity equations are discussed.

2. General formalism

In this section we collect the basic results of the Hamiltonian formalism, which will be used below for the hydrodynamic description of superfluid systems, in particular, supersolids. Thus, we are interested in the slow dynamics of a set of macroscopic field variables $\eta_\alpha(\mathbf{x}, t) \equiv \eta_\alpha(\mathbf{x})$, where index α numbers the fields. The dynamics of these fields is determined by the Hamiltonian $H[\eta]$ (the Hamiltonian is a functional of $\eta_\alpha(\mathbf{x}, t)$). In order to study the evolution of macroscopic variables, consider the following Lagrangian:

$$L = \mathfrak{L} - H = \int \mathrm{d}^3x\, F_\alpha[\mathbf{x}; \eta]\dot{\eta}_\alpha(\mathbf{x}, t) - H[\eta], \tag{1}$$

where $F_\alpha[\mathbf{x}; \eta]$ is a certain functional of $\eta_\alpha(\mathbf{x}, t)$. The first term in equation (1) represents the kinematic part of Lagrangian. The infinitesimal transformations of the fields,

$$\eta_\alpha(\mathbf{x}, t) \to \eta'_\alpha(\mathbf{x}, t) = \eta_\alpha(\mathbf{x}, t) + \delta\eta_\alpha(\mathbf{x}, t)$$

induce the variation of the action functional $W = \int_{t_1}^{t_2} \mathrm{d}t\, L$,

$$\delta W = G[t_2; \eta] - G[t_1; \eta] + \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x'\, \delta\eta_\beta(\mathbf{x}', t) \left(\int \mathrm{d}^3x\, J_{\beta\alpha}[\mathbf{x}', \mathbf{x}; \eta]\dot{\eta}_\alpha(\mathbf{x}, t) - \frac{\delta H}{\delta\eta_\beta(\mathbf{x}')} \right), \tag{2}$$

with

$$G[t; \eta] = \int \mathrm{d}^3x\, F_\alpha[\mathbf{x}; \eta]\delta\eta_\alpha(\mathbf{x}, t), \tag{3}$$

$$J_{\alpha\beta}[\mathbf{x}, \mathbf{x}'; \eta] = \frac{\delta F_\beta[\mathbf{x}'; \eta]}{\delta\eta_\alpha(\mathbf{x})} - \frac{\delta F_\alpha[\mathbf{x}; \eta]}{\delta\eta_\beta(\mathbf{x}')}. \tag{4}$$

The principle of stationary action ($\delta W = 0, \delta\eta_\alpha(\mathbf{x}, t_1) = \delta\eta_\alpha(\mathbf{x}, t_2) = 0$) gives the following equations of motion:

$$\dot{\eta}_\alpha(\mathbf{x}) = \int \mathrm{d}^3x'\, J^{-1}_{\alpha\beta}[\mathbf{x}, \mathbf{x}'; \eta]\frac{\delta H}{\delta\eta_\beta(\mathbf{x}')}, \tag{5}$$

where the inverse matrix $J^{-1}_{\alpha\beta}[\mathbf{x}, \mathbf{x}'; \eta]$ is defined by

$$\int \mathrm{d}^3x''\, J_{\alpha\gamma}[\mathbf{x}, \mathbf{x}''; \eta]J^{-1}_{\gamma\beta}[\mathbf{x}'', \mathbf{x}'; \eta] = \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{x}'). \tag{6}$$

Now we define the Poisson brackets for two arbitrary functionals $A[\eta]$ and $B[\eta]$ as follows:

$$\{A, B\} = \int \mathrm{d}^3x\, \mathrm{d}^3x'\, \frac{\delta A}{\delta\eta_\alpha(\mathbf{x})} J^{-1}_{\alpha\beta}[\mathbf{x}, \mathbf{x}'; \eta]\frac{\delta B}{\delta\eta_\beta(\mathbf{x}')}, \tag{7}$$

where $J^{-1}_{\alpha\beta}[\mathbf{x}, \mathbf{x}'; \eta] = \{\eta_\alpha(\mathbf{x}), \eta_\beta(\mathbf{x}')\}$. Then equations (5) become

$$\dot{\eta}_\alpha(\mathbf{x}) = \{\eta_\alpha(\mathbf{x}), H\} = \int \mathrm{d}^3x'\{\eta_\alpha(\mathbf{x}), \eta_\beta(\mathbf{x}')\}\frac{\delta H}{\delta\eta_\beta(\mathbf{x}')}. \tag{8}$$

Due to the antisymmetry property $J_{\alpha\beta}[\mathbf{x}, \mathbf{x}'; \eta] = -J_{\beta\alpha}[\mathbf{x}', \mathbf{x}; \eta]$, the Poisson brackets (7) meet the well-known relationships

$$\{A, B\} = -\{B, A\}, \qquad \{AB, C\} = A\{B, C\} + B\{A, C\},$$

including the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0,$$

which is satisfied in virtue of the following equality:

$$\frac{\delta J_{\alpha\beta}[\mathbf{x},\mathbf{x}';\eta]}{\delta\eta_\gamma(\mathbf{x}'')}+\frac{\delta J_{\gamma\alpha}[\mathbf{x}'',\mathbf{x};\eta]}{\delta\eta_\beta(\mathbf{x}')}+\frac{\delta J_{\beta\gamma}[\mathbf{x}',\mathbf{x}'';\eta]}{\delta\eta_\alpha(\mathbf{x})}=0.$$

Now we address the canonical transformations. To this end, consider the transformations of the form:

$$\eta_\alpha(\mathbf{x})\to\eta'_\alpha(\mathbf{x})\equiv\eta'_\alpha[\mathbf{x};\eta], \tag{9}$$

where the transformed field $\eta'_\alpha(\mathbf{x})$ is a certain functional of the initial field $\eta_\alpha(\mathbf{x})$. Those field transformations that satisfy the condition

$$\int \mathrm{d}^3x\, F_\alpha[\mathbf{x};\eta]\delta\eta_\alpha(\mathbf{x})-\int \mathrm{d}^3x\, F_\alpha[\mathbf{x};\eta']\delta\eta'_\alpha(\mathbf{x})=\delta Q[\eta] \tag{10}$$

or

$$\frac{\delta Q[\eta]}{\delta\eta_\alpha(\mathbf{x})}=F_\alpha[\mathbf{x};\eta]-\int \mathrm{d}^3x_1\, F_\beta[\mathbf{x}_1;\eta']\frac{\delta\eta'_\beta[\mathbf{x}_1;\eta]}{\delta\eta_\alpha(\mathbf{x})} \tag{11}$$

are referred to as canonical transformations. Being dependent on the structure of canonical transformations, the quantity $Q[\eta]$ represents a functional of $\eta_\alpha(\mathbf{x})$. If $Q=\mathrm{const}$, then we deal with the homogeneous canonical transformations which leave invariant the kinematic part of Lagrangian (the first term in equation (1)). Next, taking into account the identity $\delta^2Q/\delta\eta_\alpha(\mathbf{x})\delta\eta_\beta(\mathbf{x}')=\delta^2Q/\delta\eta_\beta(\mathbf{x}')\delta\eta_\alpha(\mathbf{x})$, valid for the second variational derivatives, the canonical condition (11) can be written in the form

$$J_{\alpha\beta}[\mathbf{x},\mathbf{x}';\eta]=\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\frac{\delta\eta'_\gamma[\mathbf{x}_1;\eta]}{\delta\eta_\alpha(\mathbf{x})}\frac{\delta\eta'_\lambda[\mathbf{x}_2;\eta]}{\delta\eta_\beta(\mathbf{x}')}J_{\gamma\lambda}[\mathbf{x}_1,\mathbf{x}_2;\eta']. \tag{12}$$

The Poisson brackets (8) are invariant under transformations (9) if the following condition is satisfied:

$$J^{-1}_{\alpha\beta}[\mathbf{x},\mathbf{x}';\eta']=\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\frac{\delta\eta'_\alpha[\mathbf{x};\eta]}{\delta\eta_\gamma(\mathbf{x}_1)}\frac{\delta\eta'_\beta[\mathbf{x}';\eta]}{\delta\eta_\lambda(\mathbf{x}_2)}J^{-1}_{\gamma\lambda}[\mathbf{x}_1,\mathbf{x}_2;\eta]. \tag{13}$$

It can be easily proved that equation (13) is equivalent to equation (12), which reflects the condition for transformations (9) to be canonical. Indeed, introducing the notation $T_{\alpha\beta}(\mathbf{x},\mathbf{x}')=\delta\eta'_\alpha[\mathbf{x};\eta]/\delta\eta_\beta(\mathbf{x}')$, one can write equations (12) and (13) as $\tilde{T}J[\eta']T=J[\eta]$ and $TJ^{-1}[\eta]\tilde{T}=J^{-1}[\eta']$, respectively, where $T_{\alpha\beta}(\mathbf{x},\mathbf{x}')=\tilde{T}_{\beta\alpha}(\mathbf{x}',\mathbf{x})$. Then from the first equation, we find $J^{-1}[\eta]=T^{-1}J^{-1}[\eta']\tilde{T}^{-1}$ that results in equation (13), $J^{-1}[\eta']=TJ^{-1}[\eta]\tilde{T}$.

A Lagrangian is determined up to a derivative with respect to time. It can be easily proved that $\tilde{\mathfrak{L}}_k=\tilde{\mathfrak{L}}_k+\mathrm{d}\chi(\eta)/\mathrm{d}t$ leads to the same Poisson brackets and equations of motion as $\tilde{\mathfrak{L}}_k$ (see equation (1)). In doing so, we have to take into account equations (5), (12) and (13).

We now discuss the infinitesimal canonical transformations,

$$\eta_\alpha(\mathbf{x})\to\eta'_\alpha(\mathbf{x})=\eta_\alpha(\mathbf{x})+\delta\eta_\alpha[\mathbf{x};\eta(\mathbf{x}')]. \tag{14}$$

Then in the case of homogeneous transformations ($Q=\mathrm{const}$), the canonical condition (11) reduces to

$$\int \mathrm{d}^3x'\, J_{\alpha\beta}[\mathbf{x},\mathbf{x}';\eta]\delta\eta_\beta[\mathbf{x}';\eta]=\frac{\delta G}{\delta\eta_\alpha(\mathbf{x})} \tag{15}$$

with

$$G=\int \mathrm{d}^3x'\, F_\beta[\mathbf{x}';\eta]\delta\eta_\beta[\mathbf{x}';\eta]. \tag{16}$$

Next, employing equations (6) and (7), one immediately obtains from equation (15):

$$\delta\eta_\alpha[\mathbf{x};\eta] = \{\eta_\alpha(\mathbf{x}), G\}. \tag{17}$$

Thus, the quantity G (see also equation (3)) should be interpreted as the generator of the infinitesimal canonical transformations (14). Such interpretation was given by Schwinger in the quantum action principle [41, 42]. Exactly equations (16) and (17) will be used below to find the Poisson brackets of hydrodynamic variables. Finally, note that the Hamiltonian mechanics in terms of the arbitrary dynamic variables was studied by Pauli [43] for the systems with finite degrees of freedom.

3. Thermodynamics

Normal state. The normal equilibrium state of a macroscopic system can be specified by five independent variables: the particle number density ρ, the entropy density σ and the three components of the momentum density π_k. Then, being a function of these thermodynamic variables, the energy density ε determines the equation of state, $\varepsilon = \varepsilon(\pi_k, \rho, \sigma)$. One can also choose another set of thermodynamic variables that includes the temperature T, the chemical potential μ and the velocity v_k. The corresponding thermodynamic potential density is usually denoted by ω, $\omega = \omega(T, \mu, v_k)$ ($\omega' = \omega T = -p$ is the Gibbs thermodynamic potential density and p is the pressure).

Crystalline state. The description of the states with a spontaneously broken symmetry requires the introduction of the supplementary thermodynamic variables. For a crystal, which breaks the continuous translational symmetry, such variables specify its deformation. As the elastic body is deformed, each of its initial coordinate y_i (Lagrangian coordinate) is displaced to x_i (Eulerian coordinate). These coordinates are related to the displacement vector $u_i(\mathbf{x})$,

$$x_i = y_i + u_i(\mathbf{x}). \tag{18}$$

Differentiation of equation (18) with respect to y_k gives

$$\frac{\partial x_l}{\partial y_k}\lambda_{il} = \delta_{ik}, \qquad \lambda_{il} = \delta_{il} - \frac{\partial u_i}{\partial x_l}. \tag{19}$$

The introduced quantity λ_{ik} characterizes the crystal deformation. For an ordinary crystal (without defects), the number of lattice sites coincides with the number of atoms. Since the number of lattice sites is a constant in deformed and undeformed volumes of the body, $n\delta V = n_0\delta V_0$, we come to the well-known relationship,

$$n = n_0|\lambda_{ik}|, \tag{20}$$

where n, n_0 are, respectively, the densities of lattice sites (or atoms) in the deformed and undeformed volumes δV and δV_0 of the body and $|\lambda_{ik}|$ denotes determinant of the matrix λ_{ik}. According to equation (20), the density of lattice sites n is determined by λ_{ik} and, therefore, it cannot be considered as the independent variable. Thus, the thermodynamic state of a crystal is completely specified by the three components of the momentum density π_k, the entropy density σ and λ_{ik}, so that $\varepsilon = \varepsilon(\pi_k, \sigma, \lambda_{ik})$. If the energy density ε depends on λ_{ik} only through $|\lambda_{ik}|$, then we come back to the description of the normal equilibrium state (normal liquid).

Supersolid state. As mentioned in the introduction, a supersolid state breaks both the continuous translational and the global $U(1)$ symmetries. Therefore, in order to describe the supersolid state, we have to introduce the superfluid momentum p_k (along with λ_{ik}) as the supplementary thermodynamic variable associated with $U(1)$ symmetry breaking. The

superfluid momentum is determined by the scalar potential φ, $p_k = \nabla_k \varphi$. Besides, as we have already emphasized, for the supersolid phase, the density of lattice sites n (n is determined by λ_{ik}; see equation (20)) does not coincide with the particle number density ρ. Therefore, these quantities should be considered as independent thermodynamic variables and, consequently, $\varepsilon = \varepsilon(\pi_k, \rho, \sigma, \lambda_{ik}, p_k)$.

We now introduce another set of thermodynamic variables, more useful in microscopic approach to superfluid systems [23, 25, 44]. Moreover, as we will see below (see section 8), this choice of thermodynamic variables leads to the most simple formulation of relativistic-invariant hydrodynamics of superfluid systems. Let us consider Y_a ($a = 0, k, 4$), p_k and λ_{ik} as new thermodynamic variables which define the thermodynamic potential density $\omega = \omega(Y_0, Y_k, Y_4, p_k, \lambda_{ik})$, so that

$$\mathrm{d}\omega = \varepsilon\,\mathrm{d}Y_0 + \pi_k\,\mathrm{d}Y_k + \rho\,\mathrm{d}Y_4 + \frac{\partial\omega}{\partial\lambda_{ik}}\,\mathrm{d}\lambda_{ik} + \frac{\partial\omega}{\partial p_k}\mathrm{d}p_k. \tag{21}$$

Then the densities of energy $\varepsilon \equiv \zeta_0$, momentum $\pi_k \equiv \zeta_k$ and particle number $\rho \equiv \zeta_4$ are related to ω by the following formulae:

$$\zeta_a = \frac{\partial\omega}{\partial Y_a}, \qquad a = 0, k, 4. \tag{22}$$

From equation (21), we get the basic thermodynamic identity,

$$Y_0\,\mathrm{d}\varepsilon = \mathrm{d}\sigma - Y_k\,\mathrm{d}\pi_k - Y_4\,\mathrm{d}\rho + \frac{\partial\omega}{\partial p_k}\,\mathrm{d}p_k + \frac{\partial\omega}{\partial\lambda_{ik}}\,\mathrm{d}\lambda_{ik},$$

where the entropy density is given by

$$\sigma = -\omega + Y_a\zeta_a = -\omega + Y_0\varepsilon + Y_k\pi_k + Y_4\rho. \tag{23}$$

It is easy to see that the above thermodynamic identity yields

$$\begin{aligned} &\frac{\partial\varepsilon}{\partial\sigma} = \frac{1}{Y_0} = T, \qquad \frac{\partial\varepsilon}{\partial\pi_k} = -\frac{Y_k}{Y_0} = v_k, \qquad \frac{\partial\varepsilon}{\partial\rho} = -\frac{Y_4}{Y_0} = \mu, \\ &\frac{\partial\varepsilon}{\partial p_k} = \frac{1}{Y_0}\frac{\partial\omega}{\partial p_k}, \qquad \frac{\partial\varepsilon}{\partial\lambda_{ik}} = \frac{1}{Y_0}\frac{\partial\omega}{\partial\lambda_{ik}}, \end{aligned} \tag{24}$$

where T, v_k and μ are the temperature, velocity and chemical potential, respectively.

4. Lagrangian

In non-equilibrium statistical mechanics, the locally equilibrium thermodynamic variables are assumed to describe correctly the weakly inhomogeneous states of condensed matter. Such a description is complete, i.e., there is no necessity to introduce the supplementary thermodynamic parameters in consequence of the time evolution.

In this section, we will obtain a Lagrangian describing the supersolid dynamics (hydrodynamics). The starting point is the local statistical equilibrium principle. This principle states that on hydrodynamic timescales, the thermodynamic variables are assumed to be slowly varying functions of space coordinate and time. Exactly these functions should be considered as dynamic (hydrodynamic) variables when constructing the corresponding Lagrangian. In addition, the thermodynamic energy of the system (or the Hamiltonian) is a certain functional of the local thermodynamic variables.

Consider the following Lagrangian density in terms of Lagrangian coordinate y_i:

$$L(\mathbf{y}) = \mathfrak{L}(\mathbf{y}) - \varepsilon(\mathbf{y}), \tag{25}$$

where $\mathfrak{L}(\mathbf{y})$ is the kinematic part of $L(\mathbf{y})$. The kinematic part is a linear and homogeneous function in the first time derivatives of dynamic variables. In order to construct $\mathfrak{L}(\mathbf{y})$, let us choose the following pairs of fields:

$$\pi_i(\mathbf{y},t), x_i(\mathbf{y},t), \qquad \sigma(\mathbf{y},t), \psi(\mathbf{y},t), \qquad \rho(\mathbf{y},t), \varphi(\mathbf{y},t)$$

as conjugate dynamic variables. Some of these fields, $\pi_i(\mathbf{y},t)$, $\sigma(\mathbf{y},t)$, $\rho(\mathbf{y},t)$, are the local thermodynamic variables, $\psi(\mathbf{y},t)$ is a non-physical field representing a cyclic dynamic variable (the energy density ε does not depend on it), $\varphi(\mathbf{y},t)$ is a scalar field that defines the superfluid momentum and $x_i(\mathbf{y},t)$ is Eulerian coordinate related to y_i through the displacement vector $u_i(\mathbf{x},t)$,

$$x_i(\mathbf{y},t) = y_i + u_i(\mathbf{x},t). \tag{26}$$

Note that since the formation of superfluidity breaks the global $U(1)$ symmetry generated by the conserved particle number, it is natural to take the particle number density $\rho(\mathbf{y},t)$ and the scalar field $\varphi(\mathbf{x},t)$ as conjugate variables. In view of the aforesaid, the kinematic part density $\mathfrak{L}(\mathbf{y})$ can be written in the form

$$\mathfrak{L}(\mathbf{y}) = \pi_i(\mathbf{y})\dot{x}_i(\mathbf{y}) - \sigma(\mathbf{y})\dot{\psi}(\mathbf{y}) - \rho(\mathbf{y})\dot{\varphi}(\mathbf{y}). \tag{27}$$

We see that $\mathfrak{L}(\mathbf{y})$ is given by three terms, each of which represents the kinematic part associated with one or another thermodynamic degree of freedom. In particular, the first term is similar to the kinematic part of Lagrangian in classical mechanics, $\sum p_i\dot{q}_i$

The next step is to write the Lagrangian density in terms of Eulerian coordinates x_i. To this end, we address equation (26) that gives

$$\dot{x}_i(\mathbf{y}) = \lambda^{-1}_{ik}(\mathbf{x})\dot{u}_k(\mathbf{x}) \tag{28}$$

with

$$\lambda_{ik} = \delta_{ik} - \nabla_k u_i(\mathbf{x}). \tag{29}$$

The solution $x_i = x_i(\mathbf{y},t)$ of equation (26) can be inverted, $y_i = y_i(\mathbf{x},t)$. For the sake of simplicity, we will denote Lagrangian and Eulerian fields by the same letter, e.g., $\psi(\mathbf{y},t) = \psi(\mathbf{y}(\mathbf{x},t),t) \equiv \psi(\mathbf{x},t)$. Then taking into account equation (28) and noting that

$$\dot{\psi}(\mathbf{y}) = \dot{\psi}(\mathbf{x}) + \dot{x}_i(\mathbf{y})\nabla_i\psi(\mathbf{x}), \qquad \dot{\varphi}(\mathbf{y}) = \dot{\varphi}(\mathbf{x}) + \dot{x}_i(\mathbf{y})\nabla_i\varphi(\mathbf{x}),$$

one obtains

$$\mathfrak{L}(\mathbf{x}) = \left|\frac{\partial y_k}{\partial x_l}\right| \mathfrak{L}(\mathbf{y}) = q_i(\mathbf{x})\dot{u}_i(\mathbf{x}) - \sigma(\mathbf{x})\dot{\psi}(\mathbf{x}) - \rho(\mathbf{x})\dot{\varphi}(\mathbf{x}), \tag{30}$$

where

$$q_i(\mathbf{x}) = [\pi_k(\mathbf{x}) - \sigma(\mathbf{x})\nabla_k\psi(\mathbf{x}) - \rho(\mathbf{x})\nabla_k\varphi(\mathbf{x})]\,\lambda^{-1}_{ki}(\mathbf{x}) \tag{31}$$

and

$$\pi_i(\mathbf{x}) = \left|\frac{\partial y_k}{\partial x_l}\right| \pi_i(\mathbf{y}), \qquad \sigma(\mathbf{x}) = \left|\frac{\partial y_k}{\partial x_l}\right| \sigma(\mathbf{y}), \qquad \rho(\mathbf{x}) = \left|\frac{\partial y_k}{\partial x_l}\right| \rho(\mathbf{y})$$

are the densities of momentum, entropy and particle number in Eulerian coordinates, respectively ($|\partial y_k/\partial x_l|$ denotes Jacobian of the corresponding transformation). The total Lagrangian density represents the difference between the kinematic part and the energy density of the system,

$$L(\mathbf{x}) = \mathfrak{L}(\mathbf{x}) - \varepsilon[\mathbf{x}; \pi_i(\mathbf{x}'), \rho(\mathbf{x}'), \sigma(\mathbf{x}'), p_i(\mathbf{x}'), \lambda_{ik}(\mathbf{x}')].$$

Here ε is a certain functional of the local thermodynamic variables and $p_i(\mathbf{x}) = \nabla_i\varphi(\mathbf{x})$ is the superfluid momentum. Since the energy density must be invariant under the global phase

transformations and spatial translations, it depends not on the quantities $\varphi(\mathbf{x})$, $u_i(\mathbf{x})$ but on their derivatives only, $p_i(\mathbf{x})$, $\lambda_{ik}(\mathbf{x})$. Note that our Lagrangian is a non-canonical one because $\mathfrak{L}(\mathbf{x})$ includes the nonlinear function $q_i(\mathbf{x})$ of dynamic variables. The equations of motion can be derived straightforwardly, through the principle of stationary action. However, we will use the Poisson-bracket approach to derive the dynamic equations. As we will see below, the hydrodynamic limit of these equations reproduces the non-dissipative Andreev–Lifshitz hydrodynamics of supersolids [1] (see also [21–23, 38]).

5. Poisson brackets

As we have already seen from the general formalism, the Poisson brackets can be obtained from the invariance requirement of the kinematic part of Lagrangian. Here we will study the transformations leaving invariant the kinematic part (30) and calculate the Poisson brackets for the set $\{\pi_i, u_i, \sigma, \psi, \rho, \varphi\}$ of dynamic variables. We recall that the thermodynamic energy depends on gradients of $\varphi(\mathbf{x})$ and $u_i(\mathbf{x})$ through the thermodynamic quantities $p_i(\mathbf{x}) = \nabla_i\varphi(\mathbf{x})$ and $\lambda_{ik}(\mathbf{x}) = \delta_{ik} - \nabla_k u_i(\mathbf{x})$.

We begin with the finite time-independent transformation, $x_i \to x_i' = x_i'(\mathbf{x})$. It induces the following transformation law of the displacement vector:

$$u_i(\mathbf{x}) \to u_i'(\mathbf{x}') = u_i(\mathbf{x}) + x_i' - x_i. \tag{32}$$

Equation (32) reflects the displacement of the physically infinitesimal volume with Lagrangian coordinate y_i from the point x_i to x_i'. Since $\dot{u}_i'(\mathbf{x}') = \dot{u}_i(\mathbf{x})$,

$$\lambda_{ik}(\mathbf{x}) \to \lambda_{ik}'(\mathbf{x}') = \lambda_{il}(\mathbf{x})\frac{\partial x_l}{\partial x_k'}. \tag{33}$$

If we define the following transformation properties of our dynamic variables:

$$\begin{aligned}
&\pi_i(\mathbf{x}) \to \pi_i'(\mathbf{x}') = \left|\frac{\partial x_l}{\partial x_j'}\right| \frac{\partial x_k}{\partial x_i'}\pi_k(\mathbf{x}),\\
&\sigma(\mathbf{x}) \to \sigma'(\mathbf{x}') = \left|\frac{\partial x_l}{\partial x_j'}\right| \sigma(\mathbf{x}), \qquad \psi(\mathbf{x}) \to \psi'(\mathbf{x}') = \psi(\mathbf{x}),\\
&\rho(\mathbf{x}) \to \rho'(\mathbf{x}') = \left|\frac{\partial x_l}{\partial x_j'}\right| \rho(\mathbf{x}), \qquad \varphi(\mathbf{x}) \to \varphi'(\mathbf{x}') = \varphi(\mathbf{x}),
\end{aligned} \tag{34}$$

then it is easy to show that the kinematic part (30) is invariant under the studied finite transformations. This invariance can be written in the form

$$\int \mathrm{d}^3x(q_i'(\mathbf{x})\dot{u}_i'(\mathbf{x}) - \sigma'(\mathbf{x})\dot{\psi}'(\mathbf{x}) - \rho'(\mathbf{x})\dot{\varphi}'(\mathbf{x})) = \int \mathrm{d}^3x(q_i(\mathbf{x})\dot{u}_i(\mathbf{x}) - \sigma(\mathbf{x})\dot{\psi}(\mathbf{x}) - \rho(\mathbf{x})\dot{\varphi}(\mathbf{x})).$$

Now it is evident that under the infinitesimal transformations $x_i \to x_i' = x_i + \chi_i(\mathbf{x})$ with $|\chi_i(\mathbf{x})| \ll 1$, the variations of the dynamic variables must be given by

$$\delta\eta_\alpha(\mathbf{x}) = \eta_\alpha'(\mathbf{x}) - \eta_\alpha(\mathbf{x}), \qquad \eta_\alpha = \{\pi_i, u_i, \sigma, \psi, \rho, \varphi\}.$$

Next, bearing in mind equations (32)–(34), one obtains

$$\begin{aligned}
&\delta u_i(\mathbf{x}) = \lambda_{ik}(\mathbf{x})\chi_k(\mathbf{x}), \qquad &&\delta\psi(\mathbf{x}) = -\chi_i(\mathbf{x})\nabla_i\psi(\mathbf{x}),\\
&\delta\sigma(\mathbf{x}) = -\nabla_i(\chi_i(\mathbf{x})\sigma(\mathbf{x})), \qquad &&\delta\varphi(\mathbf{x}) = -\chi_i(\mathbf{x})\nabla_i\varphi(\mathbf{x}),\\
&\delta\rho(\mathbf{x}) = -\nabla_i(\chi_i(\mathbf{x})\rho(\mathbf{x})), \qquad &&\delta\pi_i(\mathbf{x}) = -\nabla_k(\chi_k(\mathbf{x})\pi_i(\mathbf{x})) - \pi_k(\mathbf{x})\nabla_i\chi_k(\mathbf{x}).
\end{aligned} \tag{35}$$

These variations preserve invariant the kinematic part $\mathfrak{L}(\mathbf{x})$. Thus, according to the general formalism, they are infinitesimal canonical transformations with the following generator (see equation (16)):

$$G = \int \mathrm{d}^3x\,(q_i(\mathbf{x})\delta u_i(\mathbf{x}) - \sigma(\mathbf{x})\delta\psi(\mathbf{x}) - \rho(\mathbf{x})\delta\varphi(\mathbf{x})) = \int \mathrm{d}^3x\,\pi_i(\mathbf{x})\chi_i(\mathbf{x}).$$

Finally, noting that the variations (35) assume the form $\delta\eta_\alpha = \{\eta_\alpha, G\}$ (see equation (17)) and using the above explicit expression for G, one finds the Poisson brackets of $\pi_i(\mathbf{x})$ with all other dynamic variables, including the momentum density itself:

$$\begin{aligned}
&\{\pi_k(\mathbf{x}), \psi(\mathbf{x}')\} = \delta(\mathbf{x}-\mathbf{x}')\nabla_k\psi(\mathbf{x}), \qquad \{\pi_k(\mathbf{x}), \sigma(\mathbf{x}')\} = -\sigma(\mathbf{x})\nabla_k\delta(\mathbf{x}-\mathbf{x}'),\\
&\{\pi_k(\mathbf{x}), \rho(\mathbf{x}')\} = -\rho(\mathbf{x})\nabla_k\delta(\mathbf{x}-\mathbf{x}'), \qquad \{\pi_k(\mathbf{x}), \varphi(\mathbf{x}')\} = \delta(\mathbf{x}-\mathbf{x}')\nabla_k\varphi(\mathbf{x}),\\
&\{\pi_i(\mathbf{x}), \pi_k(\mathbf{x}')\} = \pi_i(\mathbf{x}')\nabla'_k\delta(\mathbf{x}-\mathbf{x}') - \pi_k(\mathbf{x})\nabla_i\delta(\mathbf{x}-\mathbf{x}'),\\
&\{\pi_i(\mathbf{x}), \lambda_{kj}(\mathbf{x}')\} = -\lambda_{ki}(\mathbf{x})\nabla_j\delta(\mathbf{x}-\mathbf{x}'),
\end{aligned} \tag{36}$$

where we have employed the fact that $\chi_i(\mathbf{x})$ is an arbitrary function.

In order to obtain the Poisson brackets missing in equations (36), we have to study another class of the infinitesimal canonical transformations. It is easy to see that the field variations

$$\begin{aligned}
&\delta q_i(\mathbf{x}) = 0, \qquad \delta u_i(\mathbf{x}) = f_i(\mathbf{x}), \qquad \delta\sigma(\mathbf{x}) = 0,\\
&\delta\psi(\mathbf{x}) = \chi(\mathbf{x}), \qquad \delta\rho(\mathbf{x}) = 0, \qquad \delta\varphi(\mathbf{x}) = \theta(\mathbf{x})
\end{aligned}$$

also leave invariant the kinematic part (30). The arbitrary real functions $f_i(\mathbf{x})$, $\chi(\mathbf{x})$ and $\theta(\mathbf{x})$ do not depend on time and dynamic variables. Following the general formalism, these variations can be written in terms of the Poisson brackets between the corresponding field and generator G (see equations (16) and (17)):

$$\begin{aligned}
&\{q_i(\mathbf{x}), G\} = 0, \qquad \{u_i(\mathbf{x}), G\} = f_i(\mathbf{x}), \qquad \{\sigma(\mathbf{x}), G\} = 0,\\
&\{\psi(\mathbf{x}), G\} = \chi(\mathbf{x}), \qquad \{\rho(\mathbf{x}), G\} = 0, \qquad \{\varphi(\mathbf{x}), G\} = \theta(\mathbf{x}),
\end{aligned}$$

with

$$G = \int \mathrm{d}^3x\,(q_i(\mathbf{x})f_i(\mathbf{x}) - \sigma(\mathbf{x})\chi(\mathbf{x}) - \rho(\mathbf{x})\theta(\mathbf{x}))\,.$$

Due to the arbitrariness of the functions $f_i(\mathbf{x})$, $\chi(\mathbf{x})$ and $\theta(\mathbf{x})$, we come to nonzero Poisson brackets, missing in equations (36),

$$\{\sigma(\mathbf{x}'), \psi(\mathbf{x})\} = \delta(\mathbf{x}-\mathbf{x}'), \qquad \{\rho(\mathbf{x}'), \varphi(\mathbf{x})\} = \delta(\mathbf{x}-\mathbf{x}') \tag{37}$$

and

$$\{u_i(\mathbf{x}), q_j(\mathbf{x}')\} = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}'). \tag{38}$$

Though the considered derivation gives also the vanishing Poisson brackets, we do not write them for the sake of brevity. Therefore, equations (36) and (37) provide all the nonzero Poisson brackets for dynamic variables $\{\pi_i, u_i, \sigma, \psi, \rho, \varphi\}$. The rest of the Poisson brackets between these variables turn to zero. In order to obtain all of them, it is also necessary to consider the kinematic part $\tilde{\mathfrak{L}}(\mathbf{x}) = -\dot{q}_i(\mathbf{x})u_i(\mathbf{x}) + \dot{\sigma}(\mathbf{x})\psi(\mathbf{x}) + \dot{\rho}(\mathbf{x})\varphi(\mathbf{x})$, which differs from equation (30) in the time derivative, and to carry out the mathematical manipulations similar to those described in this paragraph. Using the vanishing Poisson brackets and the definition of $q_i(\mathbf{x})$ (see equation (31)), it is easy to see that the bracket (38) reduces to $\{\pi_i(\mathbf{x}), \lambda_{kj}(\mathbf{x}')\}$, which was already found in equations (36). Thus, a closed system of the Poisson brackets is given by equations (36) and (37) as well as by the vanishing brackets. Up to an overall minus sign, the obtained Poisson brackets agree with the results of [26], where the superfluid ^{4}He

was studied. Note that the first brackets from equations (36) and (37) include a non-physical field $\psi(\mathbf{x})$ introduced as a dynamic variable conjugate to the entropy density $\sigma(\mathbf{x})$. We will see below that this field decouples from the dynamics.

Using the Poisson brackets (36), (37), one can compute the bracket (7) for functionals A and B of the physical dynamic variables $\{\pi_i, \lambda_{ik}, \sigma, \rho, \varphi\}$,

$$\{A, B\} = \left[\frac{\delta B}{\delta \rho}\nabla_i \rho + \frac{\delta B}{\delta \varphi}\varphi_{,i} + \frac{\delta B}{\delta \sigma}\nabla_i \sigma + \frac{\delta B}{\delta \lambda_{kj}}\nabla_j \lambda_{ki} + \frac{\delta B}{\delta \pi_k}(\pi_i \nabla_k + \nabla_i \pi_k)\right]\frac{\delta A}{\delta \pi_i} + \frac{\delta B}{\delta \pi_i}\left[\rho \nabla_i \frac{\delta A}{\delta \rho} - \varphi_{,i}\frac{\delta A}{\delta \varphi} + \sigma \nabla_i \frac{\delta A}{\delta \sigma} + \lambda_{ki}\nabla_j \frac{\delta A}{\delta \lambda_{kj}}\right] + \left\{\frac{\delta A}{\delta \rho}\frac{\delta B}{\delta \varphi} - \frac{\delta B}{\delta \rho}\frac{\delta A}{\delta \varphi}\right\},$$

where $\varphi_{,l} = \nabla_l \varphi$. Up to the terms with λ_{ik}, this result coincides with the corresponding Poisson bracket obtained in [36] for superfluid ^{4}He. The last term, in curly brackets, represents the generalized two-cocycle [36]. It comes from the Poisson bracket $\{\rho(\mathbf{x}'), \varphi(\mathbf{x})\} = \delta(\mathbf{x} - \mathbf{x}')$.

6. Hydrodynamic equations

Having obtained a closed system of the Poisson brackets, we are ready to study the corresponding equations of motion and their long-wave limit that leads to supersolid hydrodynamics. Being a functional of the local thermodynamic variables, the Hamiltonian of the system under consideration has the form

$$H = \int \mathrm{d}^3 x \varepsilon[\mathbf{x}; \pi_i(\mathbf{x}'), \rho(\mathbf{x}'), \sigma(\mathbf{x}'), p_i(\mathbf{x}'), \lambda_{ik}(\mathbf{x}')]. \tag{39}$$

The equations of motion can be easily obtained from equations (8), (36) and (37) taking into account the fact that $\psi(\mathbf{x})$ is a cyclic variable. These equations are found to be

$$\begin{gathered}
\dot{\sigma} + \nabla_i\left(\sigma \frac{\delta H}{\delta \pi_i}\right) = 0, \qquad \dot{\rho} + \nabla_i\left(\rho \frac{\delta H}{\delta \pi_i} + \frac{\delta H}{\delta p_i}\right) = 0, \\
\dot{\pi}_i + \sigma \nabla_i \frac{\delta H}{\delta \sigma} + \rho \nabla_i \frac{\delta H}{\delta \rho} + p_i \nabla_k \frac{\delta H}{\delta p_k} + \nabla_k\left(\pi_i \frac{\delta H}{\delta \pi_k}\right) + \pi_k \nabla_i \frac{\delta H}{\delta \pi_k} + \lambda_{ki}\nabla_j \frac{\delta H}{\delta \lambda_{kj}} = 0 \\
\dot{\lambda}_{ij} + \nabla_j\left(\lambda_{ik}\frac{\delta H}{\delta \pi_k}\right) = 0, \qquad \dot{p}_i + \nabla_i\left(p_k \frac{\delta H}{\delta \pi_k} + \frac{\delta H}{\delta \rho}\right) = 0, \qquad \dot{\psi} + \frac{\delta H}{\delta \pi_i}\nabla_i \psi + \frac{\delta H}{\delta \sigma} = 0,
\end{gathered} \tag{40}$$

where we have used the definition of the superfluid momentum, $p_i = \nabla_i \varphi$. We can see that ψ decouples from the dynamics of the physical fields.

We are interested in the long-wave (hydrodynamic) limit of the derived equations of motion. In this limit, equations (40) can be significantly simplified to describe the nondissipative hydrodynamics of supersolids. In order to obtain the desired hydrodynamic equations, we have to write equations (40) in the leading order in spatial gradients of the dynamic variables. It is easy to see that in zeroth order in the gradients, the variational derivatives of the Hamiltonian are replaced by the ordinary derivatives of the energy density,

$$\frac{\delta H(\eta_\alpha(\mathbf{x}'))}{\delta \eta_\alpha(\mathbf{x})} \approx \frac{\partial \varepsilon(\eta_\alpha(\mathbf{x}))}{\partial \eta_\alpha(\mathbf{x})} \equiv \frac{\partial \varepsilon}{\partial \eta_\alpha},$$

where η_α, as above, denotes the whole set of dynamic (hydrodynamic) variables. Therefore, in the hydrodynamic limit, equations (40) take the form

$$\begin{gathered}
\dot{\rho} = -\nabla_i j_i, \qquad \dot{\pi}_i = -\nabla_k t_{ik}, \qquad \dot{\sigma} = -\nabla_i g_i, \\
\dot{p}_i = -\nabla_i\left(p_k \frac{\partial \varepsilon}{\partial \pi_k} + \frac{\partial \varepsilon}{\partial \rho}\right), \qquad \dot{\lambda}_{ik} = -\nabla_k\left(\lambda_{il}\frac{\partial \varepsilon}{\partial \pi_l}\right),
\end{gathered} \tag{41}$$

where j_i is the particle number flux density, t_{ik} is the momentum flux density (or the stress tensor) and g_i is the entropy flux density,

$$j_i = \rho\frac{\partial\varepsilon}{\partial\pi_i} + \frac{\partial\varepsilon}{\partial p_i}, \qquad t_{ik} = p\delta_{ik} + \pi_i\frac{\partial\varepsilon}{\partial\pi_k} + p_i\frac{\partial\varepsilon}{\partial p_k} + \lambda_{ji}\frac{\partial\varepsilon}{\partial\lambda_{jk}}, \qquad g_i = \sigma\frac{\partial\varepsilon}{\partial\pi_i}. \tag{42}$$

The pressure p is given by

$$p = -\varepsilon + \sigma\frac{\partial\varepsilon}{\partial\sigma} + \rho\frac{\partial\varepsilon}{\partial\rho} + \pi_k\frac{\partial\varepsilon}{\partial\pi_k}. \tag{43}$$

When obtaining the equation of motion for π_i, we have employed the constraint $\operatorname{rot}\mathbf{p} = 0$, which reflects the irrotational nature of the superfluid flow, and the evident property of λ_{ik}, $\nabla_i\lambda_{kl} = \nabla_l\lambda_{ki}$ (see equation (29)). Note that the equation for λ_{ik} can be reduced to the equation for the displacement vector [21], usually used in elasticity theory,

$$\dot{u}_i - \lambda_{ij}\frac{\partial\varepsilon}{\partial\pi_j} = 0. \tag{44}$$

Equations (41)–(43) provide a complete hydrodynamic description of supersolids. The first three equations from equations (41) are the differential conservation laws for the densities of particle number, momentum and entropy. Two other equations describe the time evolution of the hydrodynamic variables related to the broken symmetries (the field ψ decouples from the dynamics). The conservation law for the entropy density gives the physical meaning of $\partial\varepsilon/\partial\pi_i$. In fact, since the superfluid flow is not accompanied by the entropy transfer, the quantity $\partial\varepsilon/\partial\pi_i = -Y_i/Y_0 \equiv v_{ni}$ should be interpreted as the normal velocity. Eulerian coordinate $x_k(\mathbf{y}, t)$ (see equation (26)) represents the lattice site position of a deformed lattice and, consequently, $\dot{x}_k(\mathbf{y}, t)$ is the lattice velocity. Making use of equations (28) and (44), we can see that $\dot{x}_i(\mathbf{y}, t) = \partial\varepsilon/\partial\pi_i = v_{ni}$. Therefore, the lattice velocity coincides with the normal velocity. Note that the obtained hydrodynamic equations do not account for any dynamic symmetry associated with Galilean or Lorentz invariance. We will study both cases below. Finally, the energy conservation law follows from equations (41)–(43). Indeed, noting that

$$\dot{\varepsilon} = \frac{\partial\varepsilon}{\partial\rho}\dot{\rho} + \frac{\partial\varepsilon}{\partial\sigma}\dot{\sigma} + \frac{\partial\varepsilon}{\partial\pi_i}\dot{\pi}_i + \frac{\partial\varepsilon}{\partial p_i}\dot{p}_i + \frac{\partial\varepsilon}{\partial\lambda_{ik}}\dot{\lambda}_{ik}$$

and using equations (41)–(43) along with the constraint $\operatorname{rot}\mathbf{p} = 0$, one obtains

$$\dot{\varepsilon} = -\nabla_k w_k, \tag{45}$$

where w_k is the energy flux density,

$$w_k = \frac{\partial\varepsilon}{\partial\pi_k}\left(\rho\frac{\partial\varepsilon}{\partial\rho} + \sigma\frac{\partial\varepsilon}{\partial\sigma} + \pi_i\frac{\partial\varepsilon}{\partial\pi_i}\right) + \frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\rho} + p_i\frac{\partial\varepsilon}{\partial\pi_i}\right) + \frac{\partial\varepsilon}{\partial\lambda_{ik}}\lambda_{ij}\frac{\partial\varepsilon}{\partial\pi_j}. \tag{46}$$

Normal liquid. Let λ_{ik} and p_i be the cyclic variables, so that $\varepsilon = \varepsilon(\pi_i, \rho, \sigma)$. Then the conservation laws for the densities of particle number, entropy and momentum become (see equations (41)–(43), (24))

$$\dot{\rho} + \nabla_i(\rho v_i) = 0, \qquad \dot{\sigma} + \nabla_i(\sigma v_i) = 0, \qquad \dot{\pi}_i + \nabla_k(p\delta_{ik} + \pi_i v_k) = 0.$$

The constraint $\pi_i = mj_i = m\rho v_i$, following from Galilean invariance (see below), allows us to transform the third equation into the Euler equation,

$$\dot{v}_i + (v_k\nabla_k)v_i = -(1/\rho m)\nabla_i p.$$

Elastic medium. Equations of elasticity can also be obtained from equations (41)–(43) if to take into account that ρ and p_i are the cyclic variables, i.e., $\varepsilon = \varepsilon(\pi_i, \nabla_i u_k, \sigma)$. For simplicity,

we restrict ourselves to the case $\sigma = 0$. Then the equations for π_i and u_i become (we use the equation for u_i instead of that, for λ_{ik})

$$\dot{u}_i = \lambda_{ik} v_k, \qquad \dot{\pi}_i = -\nabla_k \left[(-\varepsilon + \pi_k v_k)\delta_{ik} + \pi_i v_k + \lambda_{ji} \frac{\partial \varepsilon}{\partial \lambda_{jk}} \right]. \tag{47}$$

For an ordinary crystal, not superfluid and without defects, the density of lattice sites is $n = n_0|\lambda_{ik}|$. This density coincides with the particle number density ρ and satisfies the continuity equation $\dot{n} + \nabla_i(n v_i) = 0$. Therefore, in the linear order in the deviation from equilibrium, the constraint $\pi_i = mn v_i$ gives $\dot{\pi}_i = mn\ddot{u}_i$ and equation (47) itself takes the form of Cauchy's equation,

$$mn\ddot{u}_i = \nabla_k \sigma_{ik}, \tag{48}$$

where the stress tensor σ_{ik} is determined by

$$\sigma_{ik} = \frac{\partial^2 \varepsilon}{\partial u_{ls} \partial u_{ik}} u_{ls} \equiv \Lambda_{lsik} u_{ls}. \tag{49}$$

When obtaining equations (48) and (49) we have employed the fact that the energy density depends on $\nabla_i u_k$ through the symmetric strain tensor $u_{ik} = (1/2)(\nabla_i u_k + \nabla_k u_i)$. The elastic modulus tensor Λ_{lsik} possesses the following symmetry property: $\Lambda_{lsik} = \Lambda_{slik} = \Lambda_{lski} = \Lambda_{ikls}$.

Superfluid. If one consider λ_{ik} as a cyclic variable, so that $\varepsilon = \varepsilon(\pi_k, \sigma, \rho, p_k)$, then equations (41)–(43) reduce immediately to equations of nondissipative superfluid hydrodynamics (see, e.g., [26, 44]).

Vortices. The developed formalism can also be modified to describe the dynamics of vortices in a rotating ^{4}He. Indeed, proceeding from the analogy with superconductors [31], where the presence of vortices is related to a magnetic field, let us introduce a 'vector potential' **a**. Then the local parameter $\boldsymbol{\omega}$, related to vortices, represents the analog of a magnetic field of superconductors, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{a}$. In order to construct the corresponding Lagrangian, we also consider the 'electric induction vector' **d** as a dynamic variable conjugate to **a**. Now the Hamiltonian H (see equation (39)) depends on **a** through the superfluid momentum $\mathbf{p} = \boldsymbol{\nabla}\varphi - \mathbf{a}$ (rot $\mathbf{p} = -\boldsymbol{\omega}$) and **d** is a cyclic variable. The gauge-invariant kinematic part of the Lagrangian density (see equations (30) and (31)) is found to be

$$\mathfrak{L}(\mathbf{x}) = q_i(\mathbf{x})\dot{u}_i(\mathbf{x}) - \sigma(\mathbf{x})\dot{\psi}(\mathbf{x}) - \rho(\mathbf{x})\dot{\varphi}(\mathbf{x}) - d_i(\mathbf{x})\dot{a}_i(\mathbf{x}),$$

where

$$q_i(\mathbf{x}) = [\pi_k(\mathbf{x}) - \sigma(\mathbf{x})\nabla_k\psi(\mathbf{x}) - \rho(\mathbf{x})(\nabla_k\varphi(\mathbf{x}) - a_k(\mathbf{x})) - (\mathbf{d}(\mathbf{x}) \times \boldsymbol{\omega}(\mathbf{x}))_k]\lambda^{-1}_{ki}(\mathbf{x})$$

and the 'Poynting vector' $(\mathbf{d} \times \boldsymbol{\omega})$ represents the momentum density related to vortices. Two new dynamic variables evolve according to the following equations (we should remember that **d** is a cyclic variable):

$$\dot{\mathbf{d}} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{d}) - \mathbf{j}, \qquad \dot{\mathbf{a}} = \mathbf{v} \times \boldsymbol{\omega}, \tag{50}$$

where **j** is the particle number flux density (see equations (42)). Using the definition of $\boldsymbol{\omega}$ through the 'vector potential' **a**, one finds from the second equation

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}).$$

It can be shown that all equations (41)–(46), except the equation for p_i, preserve their form in the presence of vortices [45]. The evolution of p_i is governed by

$$\dot{p}_i = -\nabla_i(p_k v_k + \mu) - (\mathbf{v} \times \boldsymbol{\omega})_i. \tag{51}$$

Note that the relationship div $\mathbf{d} = \rho$ (the analog of Maxwell's equation) is consistent with the first equation from equations (50) due to the conservation law for the particle number density ρ. Thus, equations (41)–(46), in which the equation for p_i is replaced by equation (51), provide a hydrodynamic description of a superfluid (supersolid) ^{4}He with vortices (see, e.g., [26]). Both quantities ψ and $\mathbf{d}$ decouple from the dynamics.

In conclusion of this section, we rewrite the hydrodynamic equations (41)–(46) for supersolids in terms of the thermodynamic potential density $\omega = \omega(Y_0, Y_k, Y_4, \lambda_{ik}, p_k)$ (see section 3). These equations have a very compact form and they are convenient for studying both Galilean and Lorentz invariance of the system. From the thermodynamic relations (22)–(24), one can obtain the following form of the differential conservation laws for the densities of energy, momentum and particle number:

$$\dot{\zeta}_a = -\nabla_k \zeta_{ak}, \qquad a = 0, i, 4, \tag{52}$$

where $\zeta_a = \partial\omega/\partial Y_a$ ($\zeta_0 \equiv \varepsilon$, $\zeta_i = \pi_i$, $\zeta_4 \equiv \rho$) and ζ_{ak} are the corresponding flux densities,

$$\zeta_{ak} = -\frac{\partial}{\partial Y_a}\frac{\omega Y_k}{Y_0} + \frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_a}\frac{Y_4 + p_i Y_i}{Y_0} + \frac{\partial\omega}{\partial\lambda_{ik}}\frac{\partial}{\partial Y_a}\frac{\lambda_{ij}Y_j}{Y_0} \tag{53}$$

($\zeta_{0k} = w_k$ is the energy flux density, $\zeta_{ik} \equiv t_{ik}$ is the momentum flux density and $\zeta_{4k} \equiv j_k$ is the particle number flux density). Also, it is easy to see that the hydrodynamic equations for p_i and λ_{ij} reduce to

$$\dot{p}_i = \nabla_i\left(\frac{Y_4 + Y_k p_k}{Y_0}\right), \qquad \dot{\lambda}_{ik} = \nabla_k\left(\frac{\lambda_{ij}Y_j}{Y_0}\right). \tag{54}$$

Equations (52)–(54) are identical to the corresponding equations of supersolid hydrodynamics derived within the microscopic theory [23], which is based on Bogolyubov's method of quasiaverages [24]. As we will show in the following section, the requirement of Galilean invariance leads to the Andreev–Lifshitz hydrodynamics [1].

7. Galilean invariance

As we have noted, the derived hydrodynamic equations do not account for any dynamic symmetry. Here we study their invariance under Galilean transformations, $x_k \to x'_k = x_k - V_k t$, where V_k is the velocity of one inertial coordinate system with respect to the other. To this end, we address the hydrodynamic equations (52)–(54) in terms of ω, since the thermodynamic potential density ω is invariant under Galilean transformations [44],

$$\omega(Y_a, p_k, \lambda_{ik}) = \omega(Y'_a, p'_k, \lambda'_{ik}), \qquad a = 0, k, 4, \tag{55}$$

where

$$\begin{aligned} &Y_0 \to Y'_0 = Y_0, \qquad Y_k \to Y'_k = Y_k + V_k Y_0, \qquad Y_4 \to Y'_4 = Y_4 + m V_k Y_k + \frac{mV^2}{2}Y_0, \\ &p_k \to p'_k = p_k - mV_k, \qquad \lambda_{ik} \to \lambda'_{ik} = \lambda_{ik} \end{aligned} \tag{56}$$

and m is the mass of the ^{4}He atom. In equations (56), the local thermodynamic variables with prime and without it are taken at the points $\mathbf{x}'$ and $\mathbf{x}$, respectively.

It is easy to prove the consistency of equations (56) with the following transformation law for the phase φ under Galilean transformation:

$$\varphi \to \varphi' = \varphi - mV_k x_k + \frac{mV^2}{2}t. \tag{57}$$

To this end, consider the first equation from equations (54), which assumes the form

$$\dot{\varphi} = p_0, \qquad p_0 = \frac{1}{Y_0}(Y_4 + Y_k p_k). \tag{58}$$

Upon differentiating equation (57) with respect to time at fixed $\mathbf{x}'$ (the phase $\varphi(\mathbf{x})$ depends on time both explicitly and through $x_k = x'_k + V_k t$), we have $\dot{\varphi}' = \dot{\varphi} + V_k \nabla_k \varphi - (mV^2/2)$. Therefore, under Galilean transformation, p_0 is transformed according to the law:

$$p_0 \to p'_0 = p_0 + V_k p_k - \frac{mV^2}{2}.$$

The same transformation law is also obtained straightforwardly, by using the definition of p_0 through the hydrodynamic fields Y_a and equations (56). Thus, equations (56) and (57) are consistent. Finally, the transformation law for the displacement vector, $u_i \to u'_i = u_i - V_i t$, leads to the invariance of λ_{ik} under Galilean transformation.

Next, according to equations (22) and (23), the densities of energy, momentum, particle number and entropy are expressed through the thermodynamic potential density ω. Therefore, using equations (55) and (56), we come to the well-known transformation properties for these quantities under Galilean transformation,

$$\rho \to \rho = \rho, \qquad \sigma \to \sigma' = \sigma$$
$$\pi_k \to \pi'_k = \pi_k - mV_k\rho, \qquad \varepsilon \to \varepsilon' = \varepsilon - V_k\pi_k + \frac{mV^2}{2}\rho.$$

The transformation laws for the flux densities (53) have the form

$$\begin{aligned} &j_k \to j'_k = j_k - V_k\rho, \qquad t_{ik} \to t'_{ik} = t_{ik} - mV_i j_k - V_k\pi_i + mV_iV_k\rho, \\ &w_k \to w'_k = w_k - V_i t_{ik} - V_k\varepsilon + V_kV_i\pi_i + \frac{mV^2}{2}(j_k - V_k\rho). \end{aligned} \tag{59}$$

It can be easily proved that equations (52)–(54), describing the supersolid hydrodynamics, are invariant under the transformations given by equations (55), (56) and (59). In this connection, note that the hydrodynamic fields $\eta'_\alpha(\mathbf{x}')$ depend on time both explicitly and through the relation between the coordinates, $x_k = x'_k + V_k t$.

Now let us show that the constraint on the thermodynamic potential density, following from Galilean invariance, leads to the Andreev–Lifshitz supersolid hydrodynamics. To this end, consider a reference frame K' in which the superfluid momentum is zero, $p'_k = 0$. Then equation (55) takes the form

$$\omega(Y_a, p_k, \lambda_{ik}) = \omega(Y'_a, 0, \lambda'_{ik}) \tag{60}$$

and the hydrodynamic fields in the reference frames K, K' are related, according to equations (56), by the following formulae:

$$p_k = mV_k \equiv mv_{sk}, \qquad Y'_0 = Y_0, \qquad Y'_k = Y_k + v_{sk}Y_0,$$
$$Y'_4 = Y_4 + mv_{sk}Y_k + \frac{mv_s^2}{2}Y_0, \qquad \lambda'_{ik} = \lambda_{ik}.$$

where $v_{sk} \equiv v_{sk}(\mathbf{x})$ is the superfluid velocity. The flux densities (53) are expressed through the conserved quantities $\zeta_a = \partial\omega/\partial Y_a$ and $\partial\omega/\partial p_k$, $\partial\omega/\partial\lambda_{ik}$. Using equations (60) and (22), we can write these derivatives through their values in the reference frame K', where $v_{sk} = 0$,

$$\varepsilon = \varepsilon' + v_{sk}\pi'_k + \frac{mv_s^2}{2}\rho', \qquad \pi_k = \pi'_k + mv_{sk}\rho', \qquad \rho = \rho',$$
$$\frac{1}{Y_0}\frac{\partial\omega}{\partial p_k} = \frac{\pi'_k}{m} + (v_{sk} - v_{nk})\rho', \qquad \frac{\partial\omega}{\partial\lambda_{ik}} = \frac{\partial\omega}{\partial\lambda'_{ik}}.$$

Here $v_{nk} \equiv v_k = -Y_k/Y_0$ is the normal velocity that coincides with the lattice one. Therefore, the flux densities (53) assume the form

$$\begin{aligned} j_k &= \frac{\pi'_k}{m} + v_{sk}\rho, \\ t_{ik} &= [T\sigma - \varepsilon' + (v_{nl} - v_{sl})\pi'_l + \mu'\rho]\delta_{ik} + v_{nk}\pi'_i + v_{si}\pi'_k + m v_{si} v_{sk}\rho + \lambda_{ji}\frac{\partial \varepsilon}{\partial \lambda_{jk}}, \\ w_k &= v_{nk} T\sigma + v_{nk}(v_{nl}\pi'_l) + \left(\mu' + \frac{m v_s^2}{2}\right) j_k + v_{nl}\lambda_{jl}\frac{\partial \varepsilon}{\partial \lambda_{jk}}, \end{aligned} \tag{61}$$

where μ' is the chemical potential in K',

$$\mu' = \mu + m v_{nl} v_{sl} - \frac{m v_s^2}{2}. \tag{62}$$

When obtaining the expressions for t_{ik} and w_k we have employed equation (23) to eliminate the thermodynamic potential density ω (the entropy density is invariant under Galilean transformation). From the above relations, we can see the characteristic property of Galilean systems—the momentum density is proportional to the particle number flux density, $m j_k = \pi_k$. Finally, remembering the thermodynamic relations (24), we can write equations (54) in the form

$$\dot{v}_{si} + \nabla_i\left(\frac{\mu'}{m} + \frac{v_s^2}{2}\right) = 0, \qquad \dot{\lambda}_{ik} + \nabla_k(\lambda_{ij} v_{nj}) = 0. \tag{63}$$

The derived equations (61)–(63) almost coincide with the Andreev–Lifshitz equations of supersolid hydrodynamics [1] in the nondissipative limit. However, in contrast to Andreev–Lifshitz equations, the stress tensor t_{ik} includes the term nonlinear in strain (the last term).

8. Lorentz invariance

In this section we focus on the relativistic generalization of supersolid hydrodynamics. Instead of Galilean invariance, we will require relativistic invariance of a hydrodynamic theory, i.e., the invariance under the Lorentz transformations $x^\mu \to x'^\mu = a^{\mu\nu}x_\nu$, where $x^\mu = (t, \mathbf{x})$ and $a^{\mu\nu}$ $\left(a_{\rho\mu}a^{\rho\nu} = \delta^\nu_\mu\right)$ are the coefficients of the transformations.

We begin the generalization of equations (52)–(54) by introducing the following four-component quantities:

$$Y_\mu = (Y_0, Y_k), \qquad p_\mu = (p_0, p_k), \qquad \lambda_{j\mu} = (\lambda_{j0}, \lambda_{jk}),$$

where the constraints

$$p_0 = \frac{Y_4 + p_k Y_k}{Y_0}, \qquad \lambda_{j0} = \frac{\lambda_{jk} Y_k}{Y_0} \tag{64}$$

represent the right-hand side of equations (54). Equations (64) can also be written in relativistic notations,

$$Y_\mu p^\mu = Y_4, \qquad Y_\mu \lambda^\mu_j = 0. \tag{65}$$

Since the volume of the system V is not a relativistic-invariant quantity, it is natural to introduce the relativistic-invariant potential density $\omega' = \omega/Y_0$ instead of ω [44] ($\omega' = -p$ is the Gibbs potential density, where p is the pressure) and to express the conserved quantities ζ_a and their fluxes ζ_{ak} (see equations (22) and (53)) through this relativistic-invariant potential density. To this end, let us consider the function $\tilde{\omega}(Y_\mu, p_\mu, \lambda_{j\mu}) \equiv \tilde{\omega}(Y_0, Y_k; p_0, p_k; \lambda_{j0}, \lambda_{jk})$ and relate it to the Gibbs thermodynamic potential density ω':

$$\omega'(Y_0, Y_k, Y_4, p_k, \lambda_{jk}) = \tilde{\omega}(Y_0, Y_k; p_0, p_k; \lambda_{j0}, \lambda_{jk}). \tag{66}$$

Next, performing the differentiation of equation (66) with respect to all hydrodynamic fields and taking into account the constraints (64), one finds

$$\frac{\partial\omega'}{\partial Y_0}=\frac{\partial\tilde{\omega}}{\partial Y_0}-\frac{p_0}{Y_0}\frac{\partial\tilde{\omega}}{\partial p_0}-\frac{\lambda_{j0}}{Y_0}\frac{\partial\tilde{\omega}}{\partial\lambda_{j0}},\qquad \frac{\partial\omega'}{\partial Y_k}=\frac{\partial\tilde{\omega}}{\partial Y_k}+\frac{p_k}{Y_0}\frac{\partial\tilde{\omega}}{\partial p_0}+\frac{\lambda_{jk}}{Y_0}\frac{\partial\tilde{\omega}}{\partial\lambda_{j0}},$$
$$\frac{\partial\omega'}{\partial p_k}=\frac{\partial\tilde{\omega}}{\partial p_k}+\frac{Y_k}{Y_0}\frac{\partial\tilde{\omega}}{\partial p_0},\qquad \frac{\partial\omega'}{\partial Y_4}=\frac{1}{Y_0}\frac{\partial\tilde{\omega}}{\partial p_0},\qquad \frac{\partial\omega'}{\partial\lambda_{jk}}=\frac{\partial\tilde{\omega}}{\partial\lambda_{jk}}+\frac{Y_k}{Y_0}\frac{\partial\tilde{\omega}}{\partial\lambda_{j0}}. \tag{67}$$

Now, according to equations (22), the densities of conserved quantities can be written in terms of the relativistic-invariant potential density $\tilde{\omega}$:

$$\zeta_4\equiv\rho=\frac{\partial\tilde{\omega}}{\partial p_0},\qquad \zeta_0\equiv\varepsilon=\frac{\partial}{\partial Y_0}\tilde{\omega}Y_0-p_0\frac{\partial\tilde{\omega}}{\partial p_0}-\lambda_{j0}\frac{\partial\tilde{\omega}}{\partial\lambda_{j0}},$$
$$\zeta_k\equiv\pi_k=Y_0\frac{\partial\tilde{\omega}}{\partial Y_k}+p_k\frac{\partial\tilde{\omega}}{\partial p_0}+\lambda_{jk}\frac{\partial\tilde{\omega}}{\partial\lambda_{j0}}. \tag{68}$$

In a similar manner, one obtains the following formulae for the corresponding flux densities (see equation (53)):

$$\zeta_{4k}\equiv j_k=\frac{\partial\tilde{\omega}}{\partial p_k},\qquad \zeta_{0k}\equiv w_k=-Y_k\frac{\partial\tilde{\omega}}{\partial Y_0}-p_0\frac{\partial\tilde{\omega}}{\partial p_k}-\lambda_{j0}\frac{\partial\tilde{\omega}}{\partial\lambda_{jk}},$$
$$\zeta_{ik}\equiv t_{ik}=-\frac{\partial}{\partial Y_i}\tilde{\omega}Y_k+p_i\frac{\partial\tilde{\omega}}{\partial p_k}+\lambda_{ji}\frac{\partial\tilde{\omega}}{\partial\lambda_{jk}}. \tag{69}$$

Note that after performing the differentiation in equations (68) and (69), we should take into account the constraints (64). One can easily see that the particle number density $\rho\equiv j^0$ and the particle number flux density $j^k\equiv j_k$ are combined into the four-component vector $j^\mu=(\rho,\,j_k)$,

$$j^\mu=\frac{\partial\tilde{\omega}}{\partial p_\mu}. \tag{70}$$

Similarly, the densities of energy ε and momentum π_k as well as their fluxes w_k, t_{ik} form the energy–momentum tensor $t^{\mu\nu}$ of rank two,

$$t^{\mu\nu}=\frac{\partial\tilde{\omega}Y^\nu}{\partial Y_\mu}-p^\mu\frac{\partial\tilde{\omega}}{\partial p_\nu}-\lambda^\mu_j\frac{\partial\tilde{\omega}}{\partial\lambda_{j\nu}} \tag{71}$$

with $t^{00}\equiv\varepsilon$, $t^{0k}\equiv w_k$, $t^{k0}\equiv\pi_k$ and $t^{ik}\equiv t_{ik}$. Here and below, the raising and lowering of indices are implemented by means of the metric tensor $g_{\mu\nu}$ ($g_{00}=1$, $g_{ik}=-\delta_{ik}$, $g_{0k}=0$).

In terms of the introduced $t^{\mu\nu}$ and j^μ, the hydrodynamic equations (52) and (53) assume the form

$$\partial_\nu t^{\mu\nu}=0,\qquad \partial_\nu j^\nu=0, \tag{72}$$

where $\partial_\nu=\partial/\partial x^\nu$. Equations (54), describing the evolution of p_i and λ_{ik}, can also be written in relativistic notations,

$$\partial_\nu p_\mu-\partial_\mu p_\nu=0,\qquad \partial_\mu\lambda_{i\nu}-\partial_\nu\lambda_{i\mu}=0. \tag{73}$$

Note that the first equation includes the irrotational condition of the superfluid flow, $\mathrm{rot}\,\mathbf{p}=0$. The second one contains the evident property of λ_{ik}, $\nabla_k\lambda_{il}=\nabla_l\lambda_{ik}$. Finally, the entropy density evolves according to the following equation:

$$\dot{\sigma}-\nabla_i\left(\sigma\frac{Y_i}{Y_0}\right)=0,$$

which also assumes the relativistic form. Indeed, using equations (67) and (64), one can express the entropy density (23) through $\tilde{\omega}$, $\sigma\equiv\sigma^0=Y_0Y_\mu(\partial\tilde{\omega}/\partial Y_\mu)$. Then σ^0 and the

entropy flux density $\sigma^k \equiv Y_k Y_\mu(\partial\tilde{\omega}/\partial Y_\mu)$ together form a 4-vector $\sigma^\mu = (\sigma^0, \sigma^k)$ that satisfies the conservation law,

$$\partial_\mu \sigma^\mu = 0.$$

So far, we have not imposed any constraints on the thermodynamic potential density $\tilde{\omega}$ associated with relativistic invariance. In fact, we have merely changed from the independent variables Y_a $(a = 0, \ldots, 4)$, p_k, λ_{ik} to Y_μ $(\mu = 0, \ldots, 3)$, $p_\mu, \lambda_{i\mu}$, where p_0 and λ_{i0} are given by equations (64). Being a relativistic invariant [44], the potential density $\tilde{\omega}$ must be a function of invariants,

$$\tilde{\omega} = \tilde{\omega}(J_1, J_2, J_i, J_{ij}, I_1, I_j) \tag{74}$$

with

$$J_1 = (1/2)Y_\mu Y^\mu, \qquad J_2 = (1/2)p_\mu p^\mu, \qquad J_i = p_\mu \lambda_i^\mu, \qquad J_{ij} = (1/2)\lambda_{i\mu}\lambda_j^\mu,$$
$$I_1 = Y_\mu p^\mu = Y_4, \qquad I_j = Y_\mu \lambda_j^\mu.$$

Moreover, under Lorentz transformations $(x^\mu \to x'^\mu = a_\nu^\mu x^\nu)$, the hydrodynamic fields have the following transformation properties:

$$Y_\mu \to Y'_\mu = a_\mu^\nu Y_\nu, \qquad p_\mu \to p'_\mu = a_\mu^\nu p_\nu, \qquad \lambda_{j\mu} \to \lambda'_{j\mu} = a_\mu^\nu \lambda_{j\nu}.$$

Now the energy–momentum tensor (71) and the current vector (70) can be written in the final form

$$t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial\tilde{\omega}}{\partial J_1} Y^\mu Y^\nu - \frac{\partial\tilde{\omega}}{\partial J_2} p^\mu p^\nu - \frac{\partial\tilde{\omega}}{\partial J_{kj}} \lambda_k^\mu \lambda_j^\nu - \frac{\partial\tilde{\omega}}{\partial J_i}\left(p^\mu \lambda_i^\nu + p^\nu \lambda_i^\mu\right) \tag{75}$$

and

$$j^\mu = \frac{\partial\tilde{\omega}}{\partial J_2} p^\mu + \frac{\partial\tilde{\omega}}{\partial J_i} \lambda_i^\mu + \frac{\partial\tilde{\omega}}{\partial Y_4} Y^\mu, \tag{76}$$

where, after performing the necessary differentiation, we employed equations (65). It is easy to see that the energy–momentum tensor is a symmetric tensor, $t^{\mu\nu} = t^{\nu\mu}$, as it should in relativistic theory. Due to this symmetry, the momentum density coincides with the energy flux density. The obtained equations (72)–(76) provide the hydrodynamic description of a relativistic supersolid. As such object, one can consider the crystalline superfluid state that may occur in compact stars [40].

Let us consider some particular cases of equations (75) and (76).

(1) Relativistic normal liquid. The thermodynamic potential density $\tilde{\omega}$, which depends on J_1 and Y_4 only, describes the nondissipative hydrodynamics of a normal liquid,

$$t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial\tilde{\omega}}{\partial J_1} Y^\mu Y^\nu, \qquad j^\mu = \frac{\partial\tilde{\omega}}{\partial Y_4} Y^\mu.$$

These formulae also assume the form [49]:

$$t^{\mu\nu} = -p g^{\mu\nu} + w v^\mu v^\nu, \qquad j^\mu = \rho v^\mu,$$

where $v^\mu = Y^\mu/\sqrt{Y_\nu Y^\nu}$ is the 4-velocity $(v_\mu v^\mu = 1)$ and $w = (\varepsilon + p)$ is the enthalpy density. The quantities w, ε and ρ are taken in the reference frame, moving with the normal velocity v_k.

(2) Relativistic superfluid. If $\tilde{\omega}$ does not depend on J_{kj} and J_i, i.e., $\tilde{\omega} = \tilde{\omega}(J_1, J_2, Y_4)$, then equations (75) and (76) take the form

$$t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial\tilde{\omega}}{\partial J_1} Y^\mu Y^\nu - \frac{\partial\tilde{\omega}}{\partial J_2} p^\mu p^\nu, \qquad j^\mu = \frac{\partial\tilde{\omega}}{\partial J_2} p^\mu + \frac{\partial\tilde{\omega}}{\partial Y_4} Y^\mu.$$

The corresponding hydrodynamic equations, as well as the first equation from equations (73), coincide with those obtained within the microscopic approach for a relativistic superfluid [44]. There is a number of equivalent formulations of nondissipative relativistic hydrodynamics (see, e.g., [46]–[48]) in which the authors use various hydrodynamic variables. However, in terms of the chosen variables, the energy–momentum tensor contains a smaller number of relativistic invariants.

(3) Relativistic elasticity. Let $\tilde{\omega}$ be a certain function of J_1 and J_{kj} only, i.e., $\tilde{\omega} = \tilde{\omega}(J_1, J_{kj}, Y_4)$. Then equations (75) and (76) become

$$t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial \tilde{\omega}}{\partial J_1} Y^\mu Y^\nu - \frac{\partial \tilde{\omega}}{\partial J_{kj}} \lambda^\mu_k \lambda^\nu_j, \qquad j^\mu = \frac{\partial \tilde{\omega}}{\partial Y_4} Y^\mu.$$

These equations along with the second equation from equations (73) describe the relativistic elasticity.

In conclusion, we have constructed a phenomenological Lagrangian that leads to nondissipative hydrodynamics of supersolids. We show that the Poisson brackets of hydrodynamic variables are found from the invariance requirement of the kinematic part of the constructed Lagrangian. Depending on what variables are cyclic, our Lagrangian and the developed approach describe not only supersolids but also normal and superfluid liquids, as well as the elastic bodies. We also modify our Lagrangian to include the dynamics of vortices in superfluid ^{4}He. The obtained hydrodynamic equations of supersolids have the most general form—they do not account for Galilean or Lorentz invariance. The requirement of Galilean invariance gives the hydrodynamic equations, which almost coincide with Andreev–Lifshitz equations. The difference is that our stress tensor includes the term nonlinear in strain. We also present a relativistic-invariant hydrodynamic theory of supersolids, which might be useful in astrophysical applications [40]. Due to the proper choice of hydrodynamic field variables, the energy–momentum tensor has the most simple form in comparison to other relativistic theories of superfluidity (it contains a smaller number of relativistic invariants; see, e.g., [46–48]).

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